$$H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2$$

$|\varphi_n\rangle$ eigenvectores

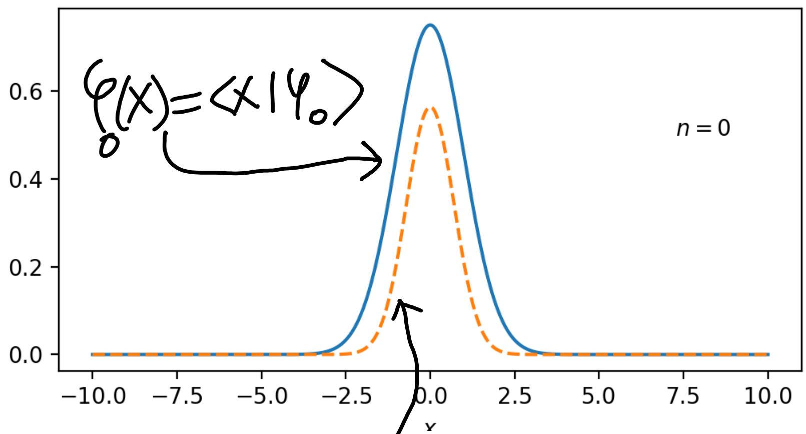
E_n eigenvalores

Operadores escalera

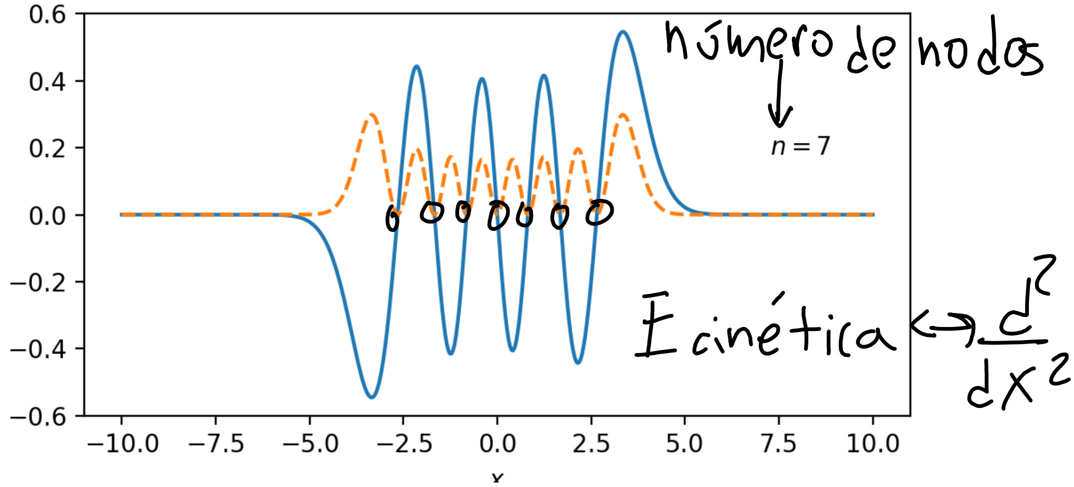
$$a|\varphi_n\rangle = \sqrt{n}\,|\varphi_{n-1}\rangle \qquad a^\dagger|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_{n+1}\rangle$$

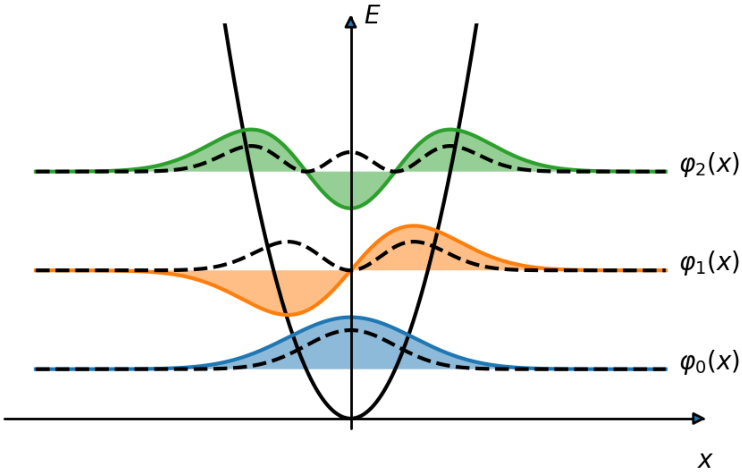
$$a|\varphi_0\rangle = 0$$

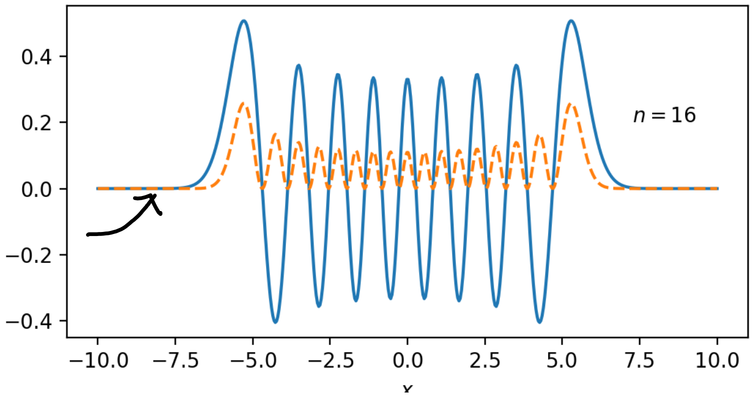
$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$



$|\varphi_0(x)|^2$







Evolución temporal:

$$|\Psi(0)\rangle = \sum_{n=0}^{\infty} C_n(0)\,|\varphi_n\rangle$$

Estado inicial general

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} C_n(0)\, e^{-iE_n t/\hbar}\,|\varphi_n\rangle$$

$$= \sum_{n=0}^{\infty} C_n(0)\, e^{-i(n+\frac{1}{2})\omega t}\,|\varphi_n\rangle$$

Estado $|\varphi_n\rangle$ oscila con frecuencia $(n+\frac{1}{2})\omega$

Si A es un observable

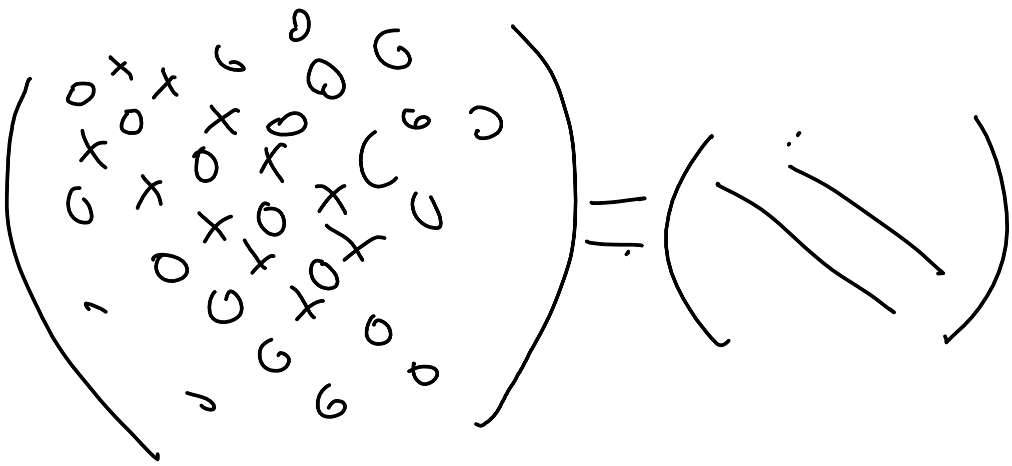
$$\langle A\rangle(t)=\langle\psi(t)|A|\psi(t)\rangle=$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_m^*(0)\,C_n(0)\,A_{mn}\,e^{-i(m-n)\omega t}$$

$$A_{mn} = \langle\varphi_m|A|\varphi_n\rangle$$

$A=X$

$$\langle\varphi_m|X|\varphi_n\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\delta_{m,n+1}+\sqrt{n}\,\delta_{m,n-1}\right]$$



P también tiene una forma similar

$\therefore$ En $\langle P\rangle(t)$ y $\langle X\rangle(t)$ sólo aparecen términos que oscilan como ωt.

¿Cómo evolucionan los valores esperados?

$$\frac{d}{dt}\langle x\rangle = \frac{1}{i\hbar}\langle [X, H]\rangle = \frac{\langle P\rangle}{m}$$

$$\frac{d}{dt}\langle P\rangle = \frac{1}{i\hbar}\langle [P, H]\rangle = -m\omega^2\langle X\rangle$$

$$\langle X\rangle(t) = \langle x\rangle(0)\cos\omega t + \frac{1}{m\omega}\langle P\rangle(0)\,\mathrm{sen}\,\omega t$$

$$\langle P\rangle(t) = \langle P\rangle(0)\cos\omega t - m\omega\langle x\rangle(0)\,\mathrm{sen}\,\omega t$$

$|\varphi_n\rangle$ edo. estacionario

Para que $\langle x\rangle(t)$ y $\langle P\rangle(t)$ cambien con t necesitamos una superposición de $|\varphi_n\rangle$.

Ejemplos de evolución temporal

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n(0) e^{-i\omega(n+\frac{1}{2})t} |\varphi_n\rangle$$

→ Si $C_0(0) = 1$ y $C_n(0) = 0 \quad n > 0$

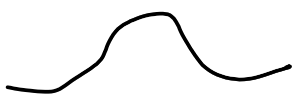
$$|\psi(t)\rangle = e^{-\frac{i\omega}{2}t} |\varphi_0\rangle$$

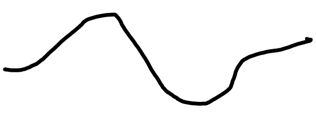
→ Si $C_{n'}(0) = 1$ y los demás $C_n(0) = 0$

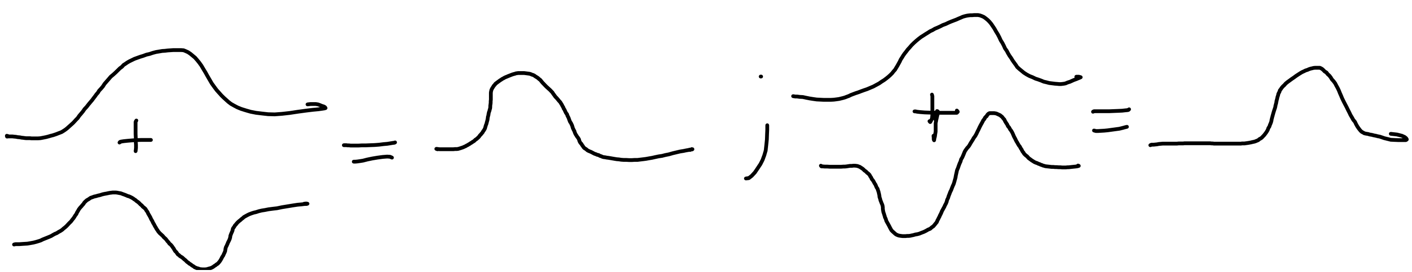
$$|\psi(t)\rangle = e^{-i\omega(n'+\frac{1}{2})t} |\varphi_{n'}\rangle$$

→ $C_0(0) = \frac{1}{\sqrt{2}}$, $C_1(0) = \frac{1}{\sqrt{2}}$, los demás son cero.

$$|\psi(t)\rangle = \frac{e^{-\frac{i\omega}{2}t}|\varphi_0\rangle + e^{-i\frac{3\omega}{2}t}|\varphi_1\rangle}{\sqrt{2}}$$

$\varphi_0 =$ 

$\varphi_1 =$ 



$$i\hbar\,\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$$

Si H no depende de t

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

$$|\psi(0)\rangle = \sum c_n(0)|\varphi_n\rangle$$

$$e^{-iHt/\hbar}|\psi(0)\rangle = \sum c_n(0)\underbrace{e^{-iHt/\hbar}|\varphi_n\rangle}_{e^{-iE_n t/\hbar}|\varphi_n\rangle}$$

$$|\psi(0)\rangle = |\varphi_0\rangle \rightarrow |\psi(t)\rangle = e^{-i\frac{\omega}{2}t}|\varphi_0\rangle$$

$$\langle\psi(t)|\psi(t)\rangle = \langle\varphi_0|\cancelto{1}{e^{\frac{i\omega t}{2}}e^{-i\frac{\omega}{2}t}}|\varphi_0\rangle = \langle\varphi_0|\varphi_0\rangle = 1$$

Cómo es $|\psi|^2$?

$$|\psi(t)\rangle = \frac{e^{-i\frac{\omega}{2}t}|\varphi_0\rangle + e^{-i\frac{3\omega}{2}t}|\varphi_1\rangle}{\sqrt{2}}$$

$$\langle\psi(t)|\psi(t)\rangle = \frac{1}{2}\left(\langle\varphi_1|e^{\frac{3i\omega t}{2}} + \langle\varphi_0|e^{\frac{i\omega t}{2}}\right)\left(e^{\frac{i\omega t}{2}}|\varphi_0\rangle + e^{\frac{3i\omega t}{2}}|\varphi_1\rangle\right)$$

$$= 1$$

$$|\psi(x,t)|^2 = \langle \psi(t) | x \rangle \langle x | \psi(t) \rangle$$

$$= \frac{1}{2} \left(\langle \varphi_1 | e^{\frac{3i\omega t}{2}} + \langle \varphi_0 | e^{\frac{i\omega t}{2}} \right) |x\rangle\langle x| \left(e^{-\frac{i\omega t}{2}} |\varphi_0\rangle + e^{-\frac{3i\omega t}{2}} |\varphi_1\rangle \right)$$

$$= \frac{1}{2} \left(\varphi_1^*(x) e^{\frac{3i\omega t}{2}} + \varphi_0^*(x) e^{\frac{i\omega t}{2}} \right) \left(e^{-\frac{i\omega t}{2}} \varphi_0(x) + e^{-\frac{3i\omega t}{2}} \varphi_1(x) \right)$$

$$= \frac{1}{2} \Big(|\varphi_0(x)|^2 + |\varphi_1(x)|^2 + \varphi_1^*(x)\varphi_0(x) e^{i\omega t} + \varphi_1(x)\varphi_0^*(x) e^{-i\omega t} \Big)$$

$$= \frac{1}{2} \left(|\varphi_0(x)|^2 + |\varphi_1(x)|^2 + R_e \{ \varphi_1^*(x)\varphi_0(x) e^{i\omega t} \} \right)$$

Cómo evoluciona $|\psi(x)|^2$ para $|\psi(0)\rangle = |\varphi_0\rangle$

$$|\psi(t)\rangle = e^{-\frac{i\omega t}{2}} |\varphi_0\rangle$$

$$|\psi(x,t)|^2 = \langle \psi(t) | x \rangle \langle x | \psi(t) \rangle$$

$$= \langle \varphi_0 | e^{+\frac{i\omega t}{2}} | x \rangle \langle x | e^{-\frac{i\omega t}{2}} | \varphi_0 \rangle$$

$$= \langle \varphi_0 | x \rangle \langle x | \varphi_0 \rangle = |\varphi_0(x)|^2$$

Estados coherentes del oscilador armónico:

son aquellos estados cuyo comportamiento se asemeja más al de un estado clásico:

- No van eigenestados de H.

$$\langle \varphi_n | X | \varphi_n \rangle = 0 \qquad \forall E_n$$
$$\langle \varphi_n | P | \varphi_n \rangle = 0$$

Oscilador clásico cumple

$$\frac{dx}{dt}(t) = \frac{1}{m} p(t) \qquad \frac{d}{dt} p(t) = -m\omega^2 X(t)$$

Para adimencionalizar $\beta = \sqrt{\frac{m\omega}{\hbar}}$ $\quad [\beta] = \frac{1}{L}$

$$\hat{X} = \beta X \qquad \hat{P} = \frac{1}{\hbar\beta} P$$

$$\frac{d}{dt}\hat{x}(t) = \omega \hat{p} \qquad \frac{d}{dt}\hat{p} = -\omega \hat{x}$$

El estado está completamente determinado por $\hat{x}$ y $\hat{p}$.

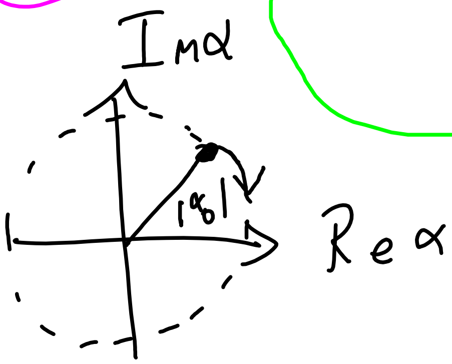
Definimos

$$\alpha(t) = \frac{1}{\sqrt{2}}\left(\hat{x}(t) + i\,\hat{p}(t)\right)$$

El estado del sistema clásico en un tiempo t está determinado por $\alpha(t)$.

$$\frac{d}{dt}\alpha(t) = -i\omega\,\alpha(t) \Longrightarrow \alpha(t) = \alpha_0 e^{-i\omega t}$$

$$\alpha_0 = \frac{1}{\sqrt{2}}\left(\hat{x}(0) + i\,\hat{p}(0)\right)$$



$$\hat{x}(t) = \frac{1}{\sqrt{2}}\left[\alpha_0 e^{-i\omega t} + \alpha_0^* e^{i\omega t}\right]$$

$$\hat{p}(t) = \frac{-i}{\sqrt{2}}\left[\alpha_0 e^{-i\omega t} - \alpha_0^* e^{i\omega t}\right]$$

$$H = cte = \frac{p^2(0)}{2m} + \frac{1}{2}m\omega^2 x^2(0) = \hbar\omega|\alpha_0|^2$$

Regresamos a la descripción cuántica.

$$\langle a\rangle(t) = \langle\psi(t)|\,a\,|\psi(t)\rangle$$

no es Hermitiano

$$i\hbar\frac{d}{dt}\langle a\rangle(t) = \langle[a,H]\rangle(t) = \hbar\omega\langle a\rangle$$

$$[a,H] = \hbar\omega[a, a^\dagger a] = \hbar\omega\, a$$

$$\frac{d}{dt}\langle a\rangle(t) = -i\omega\langle a\rangle(t)$$

$$\langle a\rangle(t) = \langle a\rangle(0)\, e^{-i\omega t}$$

$$\hat{X} = \beta X \qquad \hat{P} = \frac{1}{\hbar\beta} P$$

$$\langle\hat{x}\rangle(t) = \frac{1}{\sqrt{2}}\left[\langle a\rangle(0)\, e^{-i\omega t} + \langle a\rangle^*(0)\, e^{i\omega t}\right]$$

$$\langle\hat{p}\rangle(t) = \frac{-i}{\sqrt{2}}\left[\langle a\rangle(0)\, e^{-i\omega t} - \langle a\rangle^*(0)\, e^{+i\omega t}\right]$$

Queremos que

$$\boxed{\begin{array}{c}\text{Cuántico}\\ \langle\hat{x}\rangle(t)\\ \langle\hat{p}\rangle(t)\end{array}} \begin{array}{c}\\ =\\ =\end{array} \boxed{\begin{array}{c}\text{clásico}\\ \hat{x}(t)\\ \hat{p}(t)\end{array}}$$

Esto ocurre $\iff \langle a\rangle(0) = \alpha_0$

$$\langle\psi(0)|a|\psi(0)\rangle = \alpha_0$$

Queremos encontrar $|\psi(0)\rangle$

Por otro lado

$$\langle H\rangle = \hbar\omega\langle a^\dagger a\rangle + \cancel{\frac{\hbar\omega}{2}} \text{ ignoramos}$$

$$\langle H\rangle = \hbar\omega|\alpha_0|^2 \Rightarrow \langle\psi(0)|a^\dagger a|\psi(0)\rangle = |\alpha_0|^2$$

$\therefore$ Queremos un $|\psi(0)\rangle$ que cumpla

$$\boxed{\begin{aligned}\langle\psi(0)|a|\psi(0)\rangle &= \alpha_0\\ \langle\psi(0)|a^\dagger a|\psi(0)\rangle &= |\alpha_0|^* \\ &\quad\ \alpha_0^*\alpha_0\end{aligned}}$$

Veremos que para que esto pase $|\psi(0)\rangle$ debe ser e.V. de a.

$$b = a - \alpha_0$$

$$\langle\psi(0)|b^\dagger b|\psi(0)\rangle = \langle a^\dagger a\rangle - \alpha_0\langle a^\dagger\rangle - \alpha_0^*\langle a\rangle + \alpha_0^*\alpha_0$$

$$= \alpha_0^*\alpha_0 - \alpha_0\alpha_0^* - \alpha_0^*\alpha_0 + \alpha_0^*\alpha_0$$

$$= 0$$

$$b|\psi(0)\rangle = 0 \Rightarrow a|\psi(0)\rangle = \alpha_0|\psi(0)\rangle$$

$\therefore$ $|\psi(0)\rangle$ es un eV de a.

Vamos a usar la notación

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

Propiedades de $|\alpha\rangle$: $a|\alpha\rangle = \alpha|\alpha\rangle$

Estados coherentes

¿Cómo es $|\alpha\rangle$ en la base $\{|\varphi_n\rangle\}$?

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n(\alpha)|\varphi_n\rangle$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} C_n(\alpha)\, a|\varphi_n\rangle$$

$$= \sum_{n=1}^{\infty} C_n(\alpha)\sqrt{n}\,|\varphi_{n-1}\rangle$$

$$\overset{n \to n+1}{=} \sum_{n=0}^{\infty} C_{n+1}(\alpha)\sqrt{n+1}\,|\varphi_n\rangle$$

$$\underset{\text{Eq. e.v.}}{=} \sum_{n=0}^{\infty} \alpha\, C_n(\alpha)\,|\varphi_n\rangle$$

$$C_{n+1}(\alpha)\sqrt{n+1} = \alpha\, C_n(\alpha)$$

$$C_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}}\, C_0(\alpha)$$

Nos falta $C_0(\alpha)$.

Por normalización

$$1 = \sum_{n=0}^{\infty} |C_n|^2 = |C_0(\alpha)|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |C_0(\alpha)|^2 e^{|\alpha|^2}$$

$$\therefore \quad |C_0(\alpha)| = e^{-|\alpha|^2/2}$$

Si elegimos $C_0(\alpha)$ real y positivo

$$C_0(\alpha) = e^{-|\alpha|^2/2}$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle$$

Cuál es la probabilidad de obtener E_n al medir la energía de un sistema en el estado $|\alpha\rangle$.

$$P_n(\alpha) = |\langle \varphi_n | \alpha \rangle|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|}$$

Distribución de Poisson

$$\Delta H_\alpha^2 = \langle H^2 \rangle_\alpha - \langle H \rangle_\alpha^2 = \hbar^2\omega^2 \langle \alpha | \overbrace{(a^\dagger a + \tfrac{1}{2})^2}^{a^\dagger a a^\dagger a + a^\dagger a + \frac{1}{4}} | \alpha \rangle$$

$$- \hbar^2\omega^2 \langle \alpha | a^\dagger a + \tfrac{1}{2} | \alpha \rangle^2$$

$$= \hbar^2\omega^2 \left[\langle \alpha | a^\dagger a a^\dagger a | \alpha \rangle + |\alpha|^2 + \tfrac{1}{4} - \left(|\alpha|^2 + \tfrac{1}{2} \right)^2 \right]$$

$$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle \qquad \langle \alpha | a^\dagger = \langle \alpha | \alpha^*}$$

$$= \hbar^2\omega^2 \left[\alpha^* \langle \alpha | \underbrace{a a^\dagger}_{a^\dagger a + 1} | \alpha \rangle \alpha + \cancel{|\alpha|^2} + \cancel{\tfrac{1}{4}} - |\alpha|^4 - \cancel{|\alpha|^2} - \cancel{\tfrac{1}{4}} \right]$$

$$= \hbar^2\omega^2 \left[\cancel{|\alpha|^4} + |\alpha|^2 - \cancel{|\alpha|^4} \right] = \hbar^2\omega^2 |\alpha|^2$$

$$\Delta H_\alpha = \hbar\omega|\alpha| \qquad \langle H \rangle_\alpha = \hbar\omega \langle \alpha | a^\dagger a + \tfrac{1}{2} | \alpha \rangle$$

$$= \hbar\omega \left(|\alpha|^2 + \tfrac{1}{2} \right)$$

$$\frac{\Delta H_\alpha}{\langle H \rangle_\alpha} = \frac{\hbar\omega|\alpha|}{\hbar\omega(|\alpha|^2 + \frac{1}{2})} \sim \frac{1}{|\alpha|}$$

Cuánto vale $|\alpha|$ para una masa de 1Kg en un péndulo de $l = 10$cm con amplitud de oscilación de 1cm.

$$T = 2\pi\sqrt{\frac{l}{g}} \rightarrow \omega = \frac{2\pi}{T}$$

$$H = \hbar\omega|\alpha_0|^2 = \tfrac{1}{2} m\omega^2 x_m^2$$

$$\left\{ \alpha \sim 10^{15} \qquad \frac{\Delta H}{\langle H \rangle} \sim 10^{-15} \right.$$